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## PREDICTIONS OF NONCOMMUTATIVE SPACE-TIME

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### Abstract

An unified structure of noncommutative (NC) space-time for both gravity and particle physics is presented. This gives possibilities of testing the idea of NC space-time at the currently available energy scale. There are several arguments indicating NC space-time is visible already at the electroweak scale. This NC space-time predicts the top quark mass  $m_t \sim 172 GeV$ , the Higgs mass  $M_H \sim 241 GeV$  and the existence of a vector meson and a scalar, which interact universally with the Matter.

## 1. Introduction

Nowadays, there exist many intelligent theories, which we can neither confirm nor deny by the currently available experimental data. While using them as a tool to construct physical models, we always have an uncomfortable doubt: whether they are more fundamental than a single mathematical tool?

Noncommutative geometry (NCG) has been announced [1] as a new concept of space-time beyond the ordinary one. Its enormous and increasing influence on the whole mathematical physics suggests that NCG may have some relevance to physics. However, its achievements in particle physics [2] have not been enough to make phenomenologists believe that this sophisticated machinery contains more than putting known things together. If we want to consider more seriously the idea of noncommutative (NC) space-time and the 'dogma': *NCG is the geometry of the realistic space-time, that becomes visible at some high energy scale*, we will have to face the following questions: *i) How can we determine that energy scale ? ii) Can we test the existence of NC space-time by the currently available facilities ?* Theoretical predictions for too high energy scales would not convince an understandably impatient phenomenologist any more. Fortunately, hereafter we are able to offer some testable implications of the concept of NC space-time. The starting point is based on the idea [3] to fix the structure of NC space-time by a reasonable requirement: the physical space-time must be the same for both particle physics and general relativity. It turns out that NC space-time has a Kähler metric structure and imposes very strict constraints on the physical quantities. In fact, these constraints are valid only on a certain energy scale, because the parameters of the theory evolve independently by the quantum corrections. However, by renormalisation group arguments, we can use those constraints as initial values to determine that energy scale and derive some predictions for the currently available scale.

## 2. Noncommutative Space-time

NCG is given by the triplet  $(\mathcal{A}, D, \mathcal{H})$ , where i) the algebra  $\mathcal{A}$ , which is not necessarily commutative, generalizes the algebra  $C^\infty(\mathcal{M})$  of continuous functions on the manifold  $\mathcal{M}$  of the ordinary geometry, ii) the Dirac operator  $D$ , which generalizes the exterior derivative  $d$ , satisfies De Rham's theorem  $D^2 = 0$  and iii) the Hilbert space  $\mathcal{H}$  of the fermionic sector. The NC space-time considered in this paper is the two-copied one based on the

algebra  $C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M})$  [1-6]. A differential geometric structure of this NC space-time has been constructed by Coquereaux et al [4]. A graded structure including both even and odd differential elements has been introduced to generate a quartic Higgs potential in gauge theories. On the other hand, a theory of gravity in NCG has been also constructed using only anticommutative differential elements [5,6] in parallelism with the ordinary Einstein theory. As the space-time locally cannot be different in particle physics and in general relativity, we proposed [3] to incorporate both structures into an unique one by introducing a pair of conjugate 'discrete' differential elements instead of a real one. This is an analogue of the Newman-Penrose formalism of GR. In the Standard Model this NCG gives the same physical content as in [2,4], provided it admits a Kähler metric. Let us briefly summarize the idea.

The Hilbert space of the fermion sector  $\mathcal{H}$  is the direct sum of the Hilbert spaces  $\mathcal{H}_L$  and  $\mathcal{H}_R$  of the left- and right-handed particles:  $\mathcal{H} = \mathcal{H}_L + \mathcal{H}_R$ . Any function  $F \in \mathcal{A}$  and the Dirac operator  $D$  acting on  $\mathcal{H}$  are represented by the  $2 \times 2$  matrices

$$F = \begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix}, \quad D = d.1 + D_Q = \begin{pmatrix} d & m\theta \\ m\bar{\theta} & d \end{pmatrix} \quad (1)$$

where  $m$  is a c-number with the dimension of mass and  $\theta$  is a Clifford element.

Although our algebra  $\mathcal{A} = C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M})$  is commutative, the geometry is not, since the operator  $D$  contains a self-adjoint outer automorphism part  $D_Q$ . Here we choose  $D_Q = D_Z + D_{\bar{Z}} = DZ \partial_z + D\bar{Z} \partial_{\bar{z}}$ . The derivatives and differential elements of our geometry are given as follows

$$\begin{aligned} D_\mu &= \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix}, & DX^\mu &= \begin{pmatrix} dx^\mu & 0 \\ 0 & dx^\mu \end{pmatrix} \\ \partial_z &= \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, & DZ &= \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \\ \partial_{\bar{z}} &= \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix}, & D\bar{Z} &= \begin{pmatrix} \bar{\theta} & 0 \\ 0 & \bar{\theta} \end{pmatrix} \end{aligned} \quad (2)$$

The wedge product of any pair of differential elements is antisymmetric as in the ordinary Riemannian geometry. It is straightforward, then, to construct

geometric objects following the sample of the ordinary geometry step by step.

### 3. The Standard Model

#### 3.1 The fermionic sector and the gauge connection

In our NC space-time gauge theories can be constructed in a complete analogue of the ordinary one. It is not necessary to introduce any odd form to generate the Higgs potential. The covariant derivative is :  $\mathcal{D} = D + \mathcal{B}$ . By adjusting the fermionic sector, let us fix the gauge connection 1-form  $\mathcal{B}$ . The Lagrangian for the fermionic sector is as follows

$$\mathcal{L}_F = \langle \Psi | D + \mathcal{B} | \Psi \rangle \quad (3)$$

In the fermion sector we use the Dirac representation of the differential elements  $dx^\mu = \gamma^\mu$  and  $\theta = \gamma^5$ . The scalar product in Eq. (3) is the Clifford trace. In order to have different Higgs couplings to different types of particle we introduce the  $3N_F \times 3N_F$  mixing matrix  $M$ , where  $N_F$  is the number of generations, in the 1-form  $\mathcal{B}$  as follows

$$\mathcal{B} = \begin{pmatrix} gb_1 + 1/2.g'b_2 & \theta h \otimes M \\ \theta \bar{h} \otimes M^+ & g'b_2 \end{pmatrix} = DX^\mu B_\mu + \bar{H} \otimes M^+ D\bar{Z} + DZ H \otimes M \quad (4)$$

The gauge field and Higgs field matrices are given as follows:

$$B_\mu = \begin{pmatrix} gb_{1\mu} + 1/2.g'b_{2\mu} & 0 \\ 0 & g'b_{2\mu} \end{pmatrix} \quad , \quad H = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \quad , \quad \bar{H} = \begin{pmatrix} 0 & 0 \\ \bar{h} & 0 \end{pmatrix} \quad (5)$$

where  $b_{1\mu}$  and  $b_{2\mu}$  are respectively  $SU(2)$  and  $U(1)$  gauge connections in the Standard Model. The Higgs fields  $h$  and  $\bar{h}$  are automatically in doublets.

#### 3.2 The gauge-Higgs sector

The field strength is a direct generalization of the ordinary one

$$\Omega = D\mathcal{B} + \mathcal{B} \wedge \mathcal{B}, \quad (6)$$

$$\begin{aligned} \Omega_{\mu\nu} &= \frac{1}{2} (\partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]) = \frac{1}{2} F_{\mu\nu} \\ \Omega_{\mu z} &= \frac{1}{2} ( (\partial_\mu H + B_\mu H - H B_\mu) \otimes M - \partial_z B_\mu ) \\ \Omega_{\mu \bar{z}} &= \frac{1}{2} ( (\partial_\mu \bar{H} + B_\mu \bar{H} - \bar{H} B_\mu) \otimes M^+ - \partial_{\bar{z}} B_\mu ) \\ \Omega_{z\bar{z}} &= \frac{1}{2} (\partial_{\bar{z}} H \otimes M - \partial_z \bar{H} \otimes M^+ + \bar{H} H \otimes M^+ M - H \bar{H} \otimes M M^+) \quad (7) \end{aligned}$$

Here we choose the following Kähler metric:

$$\begin{aligned} \langle DX^\mu DX^\nu \rangle &= g^{\mu\nu} \\ \langle DX^\mu DZ \rangle &= \langle DX^\mu D\bar{Z} \rangle = 0 \\ \langle DZ D\bar{Z} \rangle &= \langle D\bar{Z} DZ \rangle = 1 \end{aligned} \quad (8)$$

The Lagrangian of the pure Yang-Mills-Higgs sector now is the generalization of the usual gauge Lagrangian  $\frac{1}{g^2} \langle F^2 \rangle$

$$\mathcal{L}_{YMH} = \frac{1}{3N_F} \text{Tr} G^{-2} \langle \Omega^2 \rangle \quad (9)$$

In order to obtain the right kinetic terms for the gauge fields, we choose the matrix  $G$  as

$$G = \begin{pmatrix} g & 0 \\ 0 & g' \sqrt{2/(2 - (g'/g)^2)} \end{pmatrix} \quad (10)$$

The factor  $1/3N_F$  is to cancel the trace of the  $3N_F \times 3N_F$  unit matrix. After shifting  $h \rightarrow h - m \otimes M^{-1}$ , we want to obtain the Lagrangian for the gauge-Higgs sector of the Standard Model

$$\mathcal{L}_{YMH} = -\frac{1}{4} \text{Tr} F^2 + \frac{1}{2} \mathcal{D}_\mu \bar{h} \mathcal{D}^\mu h + \lambda((\bar{h}h)^2 - 2v^2 \bar{h}h) + \text{constant} \quad (11)$$

In order to have the right terms in this Lagrangian, the coupling constants  $\lambda, g, g'$  and the mixing matrix  $M$  must satisfy several constraints, which can be reexpressed as the following mass formulas

$$m_h^2 = 12N_F M_W^2 \sin^2 \theta_W / (2 - \sin^2 \theta_W) \quad (12)$$

$$M_H^2 = 2m_h^2 \quad (13)$$

$$m^2 = m_h^2 \quad (14)$$

where  $m_h$  is the heaviest quark mass. Assuming three generations, our mass formulas give the top quark mass  $m_t \approx 172 \text{ GeV}$  and the Higgs mass  $M_H \approx 241 \text{ GeV}$  at the electroweak scale. The recent CDF data [7] on the top quark mass  $m_t \sim 160 - 180 \text{ GeV}$  strongly suggest that the energy scale of NC space-time is the electroweak one.

#### 4. Gravity:

In NC space-time, gravity also has new features. Next we introduce an orthonormal basis of vielbein  $\{E^A\}$  ( $A = a, z, \bar{z}$ ). As a direct generalization

of vielbein,  $E^A \doteq DX^M E_M^A$  are 1-forms in NCG. Here we are particularly interested in the self-adjoint vielbein

$$\begin{aligned} E^a &\doteq \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix} \\ E^z &\doteq \begin{pmatrix} \tilde{g} a & m \theta e^{-\kappa\sigma} \\ 0 & \tilde{g} a \end{pmatrix} \\ E^{\bar{z}} &\doteq \begin{pmatrix} \tilde{g} a & 0 \\ m \bar{\theta} e^{-\kappa\sigma} & \tilde{g} a \end{pmatrix} \end{aligned} \quad (15)$$

The vierbein  $e^a$ , the vector field  $a$  and the scalar field  $\sigma$  are real. The 'vielbein'  $E^A$  reduces to the basis  $DX^\mu, DZ$  and  $D\bar{Z}$  if the vector and scalar field vanish and the four-dimensional vierbein becomes flat. Using the Kähler flat metric in Eq. (8) for  $z, \bar{z}$  and  $\eta^{ab}$  for  $a, b = 0, 1, 2, 3$ , we can redefine the vector field  $a_\mu \rightarrow m e^{-\kappa\sigma} a_\mu$  and obtain

$$R = R_4 - 2\kappa^2 \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} \tilde{g}^2 m^2 e^{-2\kappa\sigma} f_{\mu\nu} f^{\mu\nu}, \quad (16)$$

where  $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$  and  $R_4$  is the 4-dimensional Ricci scalar.

The action is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-\det|g|} e^{-\kappa\sigma} R \quad (17)$$

Fixing the right factors of kinetic terms, we have the following formulas for the coupling constants  $\tilde{g}$  and  $\kappa$ :

$$\tilde{g} = \frac{\sqrt{16\pi G}}{m_h}, \quad \kappa = 2\sqrt{\pi G} \quad (18)$$

where  $G$  is the Newton constant, hopefully not to be confused with the gauge coupling matrix given previously. It is interesting to notice that these couplings become much stronger than the gravitational one at the scale of NC space-time.

#### 4. Discussion

The predictions of the top quark and the Higgs masses are rather attractive. They have a strong support from the recent experimental data on the top quark mass. If the Higgs particle will be found near the electroweak

scale, it would definitely support the idea of NC space-time. But, theoretically, there are still some theoretical possibilities to be clarified. It is possible, that the metric  $\langle DZ D\bar{Z} \rangle$  might depend on the coupling matrix  $G$  and/or the mixing matrix  $M$ . We have considered all these possibilities and excluded them altogether as they all have wrong physical implications. Here we do not consider the unnatural possibilities, for example, that the quark and the lepton sector might have different weight factors. The metric  $\langle DZ D\bar{Z} \rangle$ , in principle, can be any real number, however it can always be absorbed into the mass scale  $m$ . The only alternative is the number of generations  $N_F$ , which can be, of course, greater than 3. As we expect that the mass scale of the fourth generation is not as low as about  $200\text{GeV}$ , the mass formulas (12)-(14) are not valid at the electroweak scale. Given the number of generations  $N_F$ , we can determine the heaviest quark mass  $m_h$  as follows: The mass formula (12) is supposed to be valid at the scale  $m = m_h$  of NC space time, hence

$$m = \sqrt{12N_F M_W(m) \sin\theta_W(m)} \sqrt{(2 - \sin^2\theta_W(m))} \quad (19)$$

As the right-hand side of this equation will evolve, using the values of  $M_W$  and  $\theta_W$  at the electroweak scale as initial values we can determine the intersection point of its curve with the line  $y = m$  and hence the scale of NC space-time  $m = m_h$ . Having that scale we can use the mass formula (13) to determine the Higgs mass.

If we will not find the Higgs particle at an energy scale as low as  $241\text{GeV}$  in the near future, we really have to consider the possibilities of NC space-time with the number of generations  $N_F > 3$ . So, a confirmation of NC space-time from the Standard Model is waiting for the observation of the Higgs particles. Fortunately, the NC gravity can give an independent prediction for the heaviest quark mass  $m_h$  through Eq. (18). We can couple the NC gravity to the fermionic sector of the Standard Model and obtain the following interaction terms

$$\mathcal{L}_{f-a} = 4\sqrt{\pi G} \bar{\Psi} a^\mu \gamma_\mu \gamma^5 \Psi \quad (20)$$

$$\mathcal{L}_{f-\sigma} = -4m_h \sqrt{\pi G} \bar{\Psi} \gamma^5 \sigma \Psi \quad (21)$$

The couplings of the vector field  $a$  and the scalar field  $\sigma$  to the matter fields are universal and much stronger than the gravitational one. So, their existence should be testable in cosmology.

Although, all these problems must be worked out in details, it is remarkable that the idea of NC space-time can be certainly confirmed or denied by the currently available facilities. Works to answer the remaining questions are under progress.

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